

# On the structure of formal balls of the balanced quasi-metric domain of words

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## Abstract

In “Denotational semantics for programming languages, balanced quasi-metrics and fixed points” (International Journal of Computer Mathematics 85 (2008), 623-630), J. Rodríguez-López, S. Romaguera and O. Valero introduced and studied a balanced quasi-metric on any domain of (finite and infinite) words, denoted by  $q_b$ . In this paper we show that the poset of formal balls associated to  $q_b$  has the structure of a continuous domain.

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## 1 Introduction and preliminaries

Throughout this paper the symbols  $\mathbb{R}^+$  and  $\mathbb{N}$  will denote the set of all non-negative real numbers and the set of all positive integer numbers, respectively.

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Our basic references for quasi-metric spaces are [5, 11], for general topology it is [8] and for domain theory is [9].

In 1998, Edalat and Heckmann [7] established an elegant connection between the theory of metric spaces and domain theory by means of the notion of a (closed) formal ball.

Let us recall that a formal ball for a set  $X$  is simply a pair  $(x, r)$ , where  $x \in X$  and  $r \in \mathbb{R}^+$ . The set of all formal balls for  $X$  is denoted by  $\mathbf{B}X$ .

Edalat and Heckmann observed that, given a metric space  $(X, d)$ , the relation  $\sqsubseteq_d$  defined on  $\mathbf{B}X$  as

$$(x, r) \sqsubseteq_d (y, s) \Leftrightarrow d(x, y) \leq r - s,$$

for all  $(x, r), (y, s) \in \mathbf{B}X$ , is a partial order on  $\mathbf{B}X$ . Thus  $(\mathbf{B}X, \sqsubseteq_d)$  is a poset. In particular, they proved the following.

**Theorem 1** ([7]). *For a metric space  $(X, d)$  the following are equivalent:*

- (1)  $(X, d)$  is complete.
- (2)  $(\mathbf{B}X, \sqsubseteq_d)$  is a dcpo.
- (3)  $(\mathbf{B}X, \sqsubseteq_d)$  is a continuous domain.

Later on, Aliakbari et al. [1], and Romaguera and Valero [20] studied the extension of Edalat-Heckmann's theory to the framework of quasi-metric spaces.

Let us recall that a quasi-metric space is a pair  $(X, d)$  where  $X$  is a set and  $d : X \times X \rightarrow \mathbb{R}^+$  satisfies the following conditions for all  $x, y, z \in X$  :

- (i)  $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$ ;
- (ii)  $d(x, y) \leq d(x, z) + d(z, y)$ .

The function  $d$  is said to be a quasi-metric on  $X$ .

If the quasi-metric  $d$  satisfies for all  $x, y \in X$  the condition

- (i')  $x = y \Leftrightarrow d(x, y) = 0$ ,

then  $d$  is called a  $T_1$  quasi-metric and the pair  $(X, d)$  is said to be a  $T_1$  quasi-metric space.

If  $d$  is a quasi-metric on a set  $X$ , then function  $d^s$  defined as  $d^s(x, y) = \max\{d(x, y), d(y, x)\}$  for all  $x, y \in X$ , is a metric on  $X$ .

Next we recall some notions and properties of domain theory which will be useful later on.

A partially ordered set, or poset for short, is a (non-empty) set  $X$  equipped with a (partial) order  $\sqsubseteq$ . It will be denoted by  $(X, \sqsubseteq)$  or simply by  $X$  if no confusion arises.

A subset  $D$  of a poset  $X$  is directed provided that it is non-empty and every finite subset of  $D$  has upper bound in  $D$ .

A poset  $X$  is said to be directed complete, and is called a dcpo, if every directed subset of  $X$  has a least upper bound. The least upper bound of a subset  $D$  of  $X$  is denoted by  $\sqcup D$  if it exists.

Let  $X$  be a poset and  $x, y \in X$ ; we say that  $x$  is way below  $y$ , in symbols  $x \ll y$ , if for each directed subset  $D$  of  $X$  having least upper bound  $\sqcup D$ , the relation  $y \sqsubseteq \sqcup D$  implies the existence of some  $u \in D$  with  $x \sqsubseteq u$ .

A poset  $X$  is called continuous if for each  $x \in X$ , the set  $\Downarrow x := \{y \in X : y \ll x\}$  is directed with least upper bound  $x$ .

A continuous poset which is also a dcpo is called a continuous domain or, simply, a domain.

In the sequel we shall denote by  $\Sigma$  a non-empty alphabet and by  $\Sigma^\infty$  the set of all finite and infinite words (or strings) on  $\Sigma$ . We assume that the empty word  $\phi$  is an element of  $\Sigma^\infty$ , and denote by  $\sqsubseteq$  the prefix order on  $\Sigma^\infty$ . In particular, if  $x \sqsubseteq y$  and  $x \neq y$ , we write  $x \sqsubset y$ . For each  $x, y \in \Sigma^\infty$  we denote by  $x \sqcap y$  the longest common prefix of  $x$  and  $y$ , and for each  $x \in \Sigma^\infty$  we denote by  $\ell(x)$  the length of  $x$ . In particular,  $\ell(\phi) = 0$ .

It is well known that  $\Sigma^\infty$  endowed with the prefix order has the structure of a domain.

Usually it is defined a distinguished complete metric  $d_B$  on  $\Sigma^\infty$ , the so-called Baire metric (or Baire distance), which is given by

$d_B(x, x) = 0$  for all  $x \in \Sigma^\infty$ , and  $d_B(x, y) = 2^{-\ell(x \sqcap y)}$  for all  $x, y \in \Sigma^\infty$  with  $x \neq y$ .

(We adopt the convention that  $2^{-\infty} = 0$ ).

Observe that  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{d_B})$  is also a domain by Theorem 1 above.

Recall that the classical Baire metric (or Baire distance) provides a suitable framework to obtain denotational models for programming languages and parallel computation [2, 3, 4, 10] as well as to study the representation of real numbers by means of regular languages [14]. However, the Baire metric is not able to decide if a word  $x$  is a prefix of another word  $y$ , or not, in general. In order to avoid this disadvantage, some interesting and useful quasi-metric modifications of the Baire metric has been constructed. For instance:

- (A) The quasi-metric  $d_w$  defined on  $\Sigma^\infty$  as (compare [12, 15, 18, 20, etc.])  
 $d_w(x, y) = 2^{-\ell(x \sqcap y)} - 2^{-\ell(x)}$  for all  $x, y \in \Sigma^\infty$ .
- (B) The quasi-metric  $d_0$  defined on  $\Sigma^\infty$  as (compare [12, 16, 20, 21, etc.])  
 $d_0(x, y) = 0$  if  $x$  is a prefix of  $y$ ,  
 $d_0(x, y) = 2^{-\ell(x \sqcap y)}$  otherwise.
- (C) The  $T_1$  quasi-metric  $q_b$  defined on  $\Sigma^\infty$  as (compare [16])  
 $q_b(x, y) = 2^{-\ell(x)} - 2^{-\ell(y)}$  if  $x$  is a prefix of  $y$ ,  
 $q_b(x, y) = 1$  otherwise.

Observe that in Examples (A) and (B) above, the fact that a word  $x$  is a prefix of another word  $y$  is equivalent to say that the distance from  $x$  to  $y$  is exactly zero, so this condition can be used to distinguish between the case that  $x$  is a prefix of  $y$  and the remaining cases for  $x, y \in \Sigma^\infty$ . Observe also that  $(d_0)^s$  coincides with the Baire metric while  $(d_w)^s$  does not.

Nevertheless, if  $x, y, z \in \Sigma^\infty$  satisfy  $x \sqsubset y \sqsubset z$ , one obtains  $d_w(x, z) = d_w(y, z) = d_0(x, z) = d_0(y, z) = 0$ , and it is not possible to decide which word of the two,  $x$  or  $y$ , provides a better approximation to  $z$ . The quasi-metric  $q_b$  as constructed in (C) saves this inconvenience because if  $x \sqsubset y \sqsubset z$ , it follows that  $\ell(x) < \ell(y) < \ell(z)$ , and thus  $q_b(y, z) < q_b(x, z)$ . Moreover, for  $x \neq \phi$ ,  $x$  is a prefix of  $y$  if and only if  $q_b(x, y) < 1$ , so this condition also allows us to distinguish between the case that  $x$  is a prefix of  $y$  and the rest of cases (see [16, Remark 3]). We also point out that, contrarily to  $d_w$  and  $d_0$ , the quasi-metric  $q_b$  has rich topological and distance properties; in particular, it is a balanced quasi-metric in the sense of Doitchinov [6], and consequently its induced topology is Hausdorff and completely regular [16, Theorem 1 and Remark 4].

By [19, Theorem 3.1] (see also [20, p. 461]),  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{d_w})$  is a domain. On the other hand, it was shown in [20, Example 3.1] that  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{d_0})$  is a domain. In the light of these results, it seems natural to wonder if  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  is also a domain. Here we show that, indeed, this is the case.

## 2 The results

In the rest of the paper, given a quasi-metric space  $(X, d)$ , the way below relation associated to  $\sqsubseteq_d$  will be denoted by  $\ll_d$ .

**Lemma 1** ([1]). *For any quasi-metric space  $(X, d)$  the following holds:*

$$(x, r) \ll_d (y, s) \Rightarrow d(x, y) < r - s.$$

**Lemma 2.** *Let  $(X, d)$  be a quasi-metric space. If there is  $(x, r) \in \mathbf{BX}$  such that  $(x, r + s) \ll_d (x, r)$  for all  $s > 0$ , then  $\Downarrow (x, r)$  is directed and  $(x, r) = \sqcup \Downarrow (x, r)$ .*

*Proof.* Obviously  $\Downarrow (x, r) \neq \emptyset$ . Now let  $(y, s), (z, t) \in \mathbf{BX}$  such that  $(y, s) \ll_d (x, r)$  and  $(z, t) \ll_d (x, r)$ . By Lemma 1,  $d(y, x) < s - r - \varepsilon$  and  $d(z, x) < t - r - \varepsilon$  for some  $\varepsilon > 0$ . Thus  $(y, s) \sqsubseteq_d (x, r + \varepsilon)$  and  $(z, t) \sqsubseteq_d (x, r + \varepsilon)$ . Since  $(x, r + \varepsilon) \in \Downarrow (x, r)$ , we conclude that  $\Downarrow (x, r)$  is directed.

Finally, let  $(z, t)$  be an upper bound of  $\Downarrow (x, r)$ . In particular, we have that  $(x, r + 1/n) \sqsubseteq_d (z, t)$  for all  $n$ , so  $d(x, z) \leq r - t + 1/n$  for all  $n$ . Hence  $d(x, z) \leq r - t$ , i.e.,  $(x, r) \sqsubseteq_d (z, t)$ . Consequently  $(x, r) = \sqcup \Downarrow (x, r)$ .

A net  $(x_\alpha)_{\alpha \in \Lambda}$  in a quasi-metric space  $(X, d)$  is called left K-Cauchy [17, 22] (or simply, Cauchy [13]) if for each  $\varepsilon > 0$  there is  $\alpha_\varepsilon \in \Lambda$  such that  $d(x_\alpha, x_\beta) < \varepsilon$  whenever  $\alpha_\varepsilon \leq \alpha \leq \beta$ . The notion of a left K-Cauchy sequence is defined in the obvious manner.

Let  $(X, d)$  be a quasi-metric space. An element  $x \in X$  is said to be a Yoneda-limit of a net  $(x_\alpha)_{\alpha \in \Lambda}$  in  $X$  if for each  $y \in X$ , we have  $d(x, y) = \inf_\alpha \sup_{\beta \geq \alpha} d(x_\beta, y)$ . Recall that the Yoneda-limit of a net is unique if it exists.

A quasi-metric space  $(X, d)$  is called Yoneda-complete if every left K-Cauchy net in  $(X, d)$  has a Yoneda-limit, and it is called sequentially Yoneda-complete if every left K-Cauchy sequence in  $(X, d)$  has a Yoneda-limit.

**Lemma 3** ([20, Proposition 2.2]). *A  $T_1$  quasi-metric space is Yoneda-complete if and only if it is sequentially Yoneda-complete.*

**Proposition 1.** *The quasi-metric space  $(\Sigma^\infty, q_b)$  is Yoneda-complete.*

*Proof.* Since  $(\Sigma^\infty, q_b)$  is a  $T_1$  quasi-metric space it suffices to show, by Lemma 3, that it is sequentially Yoneda-complete. To this end, let  $(x_n)_{n \in \mathbb{N}}$  be a left K-Cauchy sequence in  $(\Sigma^\infty, q_b)$ . Then, there is  $n_1 \in \mathbb{N}$  such that  $q_b(x_n, x_m) < 1$  whenever  $n_1 \leq n \leq m$ . So,  $x_n$  is a prefix of  $x_m$ , i.e.,  $x_n \sqsubseteq x_m$ , whenever  $n_1 \leq n \leq m$ .

Now we distinguish two cases.

Case 1. There exists  $n_0 \geq n_1$  such that  $x_n = x_{n_0}$  for all  $n \geq n_0$ . Then, it is clear that

$$q_b(x_{n_0}, y) = \inf_n \sup_{m \geq n} q_b(x_m, y).$$

for all  $y \in X$ .

Case 2. For each  $n \geq n_1$  there exists  $m > n$  such that  $x_n \sqsubset x_m$ . In this case, there exists  $x = \sqcup \{x_n : n \geq n_1\}$ , and  $\ell(x) = \infty$ . We shall show that  $x$  is the Yoneda-limit of the sequence  $(x_n)_{n \in \mathbb{N}}$ .

Indeed, we first note that  $q_b(x_n, x) = 2^{-\ell(x_n)}$  for all  $n \geq n_1$ , and hence

$$\sup_{m \geq n} q_b(x_m, x) = \sup_{m \geq n} 2^{-\ell(x_m)} = 2^{-\ell(x_n)},$$

whenever  $n \geq n_1$ . Therefore

$$\inf_n \sup_{m \geq n} q_b(x_m, x) = \inf_n 2^{-\ell(x_n)} = 0 = q_b(x, x).$$

Finally, let  $y \in \Sigma^\infty$  such that  $y \neq x$ . Since  $\ell(x) = \infty$  it follows that  $x$  is not a prefix of  $y$ , and thus for each  $n \in \mathbb{N}$  there exists  $m \geq \max\{n, n_1\}$  such that  $x_m$  is not a prefix of  $y$ , so  $q_b(x_m, y) = 1$ . We conclude that

$$\inf_n \sup_{m \geq n} q_b(x_m, y) = 1 = q_b(x, y).$$

This finishes the proof.

**Lemma 4** ([1]). *Let  $(X, d)$  be a quasi-metric space.*

(a) *If  $D$  is a directed subset of  $\mathbf{B}X$ , then  $(y_{(y,r)})_{(y,r) \in D}$  is a left  $K$ -Cauchy net in  $(X, d)$ .*

(b) *If  $\mathbf{B}X$  is a dcpo and  $D$  is a directed subset of  $\mathbf{B}X$  having least upper bound  $(z, s)$ , then  $s = \inf\{r : (y, r) \in D\}$  and  $z$  is the Yoneda-limit of the net  $(y_{(y,r)})_{(y,r) \in D}$ .*

(c) *If  $(X, d)$  is Yoneda-complete, the poset  $(\mathbf{B}X, \sqsubseteq_d)$  is a dcpo.*

**Proposition 2.** *For each  $x \in \Sigma^\infty$  such that  $\ell(x) < \infty$ , each  $u \in \mathbb{R}^+$  and each  $v > 0$ , we have*

$$(x, u + v) \ll_{q_b} (x, u).$$

*Proof.* Let  $x \in \Sigma^\infty$  with  $\ell(x) < \infty$ ,  $u \in \mathbb{R}^+$  and  $v > 0$ , and let  $D$  be a directed subset of  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  whose least upper bound  $(z, s)$  satisfies

$(x, u) \sqsubseteq_{q_b} (z, s)$ . (The existence of least upper bound is guaranteed by Proposition 1 and Lemma 4(c)). We shall show that there exists  $(y, r) \in D$  such that  $(x, u + v) \sqsubseteq_{q_b} (y, r)$ .

We first note that, by Lemma 4 (a), there exists  $(y_1, r_1) \in D$  such that  $q_b(y_{(y,r)}, y'_{(y',r')}) < 1$  whenever  $(y, r), (y', r') \in D$  with  $(y_1, r_1) \sqsubseteq_{q_b} (y, r) \sqsubseteq_{q_b} (y', r')$ . Therefore, by the definition of  $q_b$ , we deduce that  $y_{(y,r)}$  is a prefix of  $y'_{(y',r')}$  whenever  $(y_1, r_1) \sqsubseteq_{q_b} (y, r) \sqsubseteq_{q_b} (y', r')$ .

Furthermore, by Lemma 4 (b), we have  $s = \inf\{r : (y, r) \in D\}$ , and there exists  $(y_0, r_0) \in D$ , with  $(y_1, r_1) \sqsubseteq_{q_b} (y_0, r_0)$ , such that  $y_{(y,r)}$  is a prefix of  $z$  whenever  $(y_0, r_0) \sqsubseteq_{q_b} (y, r)$ .

Now we distinguish two cases.

Case 1.  $x$  is a prefix of  $z$ . Since, by assumption,  $\ell(x) < \infty$ , there exists  $(y, r) \in D$  such that  $(y_0, r_0) \sqsubseteq_{q_b} (y, r)$ ,  $r < s + v$ , and  $x$  is a prefix of  $y_{(y,r)}$ . Then

$$q_b(x, y_{(y,r)}) = 2^{-\ell(x)} - 2^{-\ell(y_{(y,r)})} \leq 2^{-\ell(x)} - 2^{-\ell(z)} = q_b(x, z) \leq u - s < u + v - r,$$

and hence  $(x, u + v) \sqsubseteq_{q_b} (y, r)$ .

Case 2.  $x$  is not a prefix of  $z$ . Since, by assumption,  $(x, u) \sqsubseteq_{q_b} (z, s)$ , we deduce that  $q_b(x, z) = 1 \leq u - s$ . Choose  $(y, r) \in D$  such that  $r < s + v$ . Then

$$q_b(x, y_{(y,r)}) \leq 1 \leq u - s < u + v - r,$$

and hence  $(x, u + v) \sqsubseteq_{q_b} (y, r)$ . The proof is complete.

**Theorem.** *The poset of formal balls  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  is a domain.*

*Proof.* From Proposition 1 and Lemma 4 (c) it follows that the poset  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  is a dcpo, so it is only necessary to prove that it is also a continuous poset.

To this end we distinguish two cases.

Case 1. Let  $(x, r) \in \mathbf{B}\Sigma^\infty$  such that  $\ell(x) < \infty$ . By Proposition 2 and Lemma 2,  $\Downarrow (x, r)$  is a directed subset of  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  for which  $(x, r)$  is its least upper bound.

Case 2. Let  $(x, r) \in \mathbf{B}\Sigma^\infty$  be such that  $\ell(x) = \infty$ . Choose a sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of  $\Sigma^\infty$  such that  $\ell(x_n) = n$ ,  $x_n \sqsubset x_{n+1}$  and  $x_n \sqsubset x$  for all  $n \in \mathbb{N}$ . By Lemma 4 (a),  $(x_n)_{n \in \mathbb{N}}$  is a left K-Cauchy sequence, of distinct elements, in  $(\Sigma^\infty, q_b)$ , and, by Lemma 4 (b),  $x$  is its Yoneda-limit.

Similarly to the proof of Proposition 2 we shall show that  $(x_n, 2^{-n} + r) \ll_{q_b} (x, r)$  for all  $n \in \mathbb{N}$ , which implies, in particular, that  $\Downarrow (x, r) \neq \emptyset$ .

Indeed, let  $D$  be a directed subset of  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  with least upper bound  $(z, t)$  such that  $(x, r) \sqsubseteq_{q_b} (z, t)$ . Then  $t \leq r$ , and, by Lemma 4 (b),  $t = \inf\{s : (y, s) \in D\}$ , and there exists  $(y_0, s_0) \in D$  such that  $y_{(y,s)}$  is a prefix of  $z$  whenever  $(y_0, s_0) \sqsubseteq_{q_b} (y, s)$ .

If  $x = z$ , from the fact that  $x_n$  is a prefix of  $x$  we deduce the existence of some  $(y, s) \in D$  such that  $(y_0, s_0) \sqsubseteq_{q_b} (y, s)$ ,  $s < t + 2^{-\ell(y_{(y,s)})}$ , and  $x$  is a prefix of  $y_{(y,s)}$ . Therefore

$$q_b(x_n, y_{(y,s)}) = 2^{-n} - 2^{-\ell(y_{(y,s)})} \leq 2^{-n} + t - s \leq 2^{-n} + r - s,$$

so that  $(x_n, 2^{-n} + r) \sqsubseteq_{q_b} (y, s)$ .

If  $x \neq z$  we have  $q_b(x, z) = 1 \leq r - t$ . Let  $(y, s) \in D$  such that  $s < t + 2^{-n}$ . Thus

$$q_b(x_n, y_{(y,s)}) \leq 1 \leq r - t < r + 2^{-n} - s,$$

so that  $(x_n, 2^{-n} + r) \sqsubseteq_{q_b} (y, s)$ .

Next we show that  $\Downarrow (x, r)$  is directed. Indeed, let  $(y, s), (z, t) \in \mathbf{B}\Sigma^\infty$  be such that  $(y, s) \ll_{q_b} (x, r)$  and  $(z, t) \ll_{q_b} (x, r)$ . Since  $((x_n, 2^{-n} + r))_{n \in \mathbb{N}}$  is an ascending sequence in  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  with least upper bound  $(x, r)$ , there exists  $k \in \mathbb{N}$  such that  $(x_k, 2^{-k} + r)$  is an upper bound of  $(y, s)$  and  $(z, t)$ . From the fact, proved above, that  $(x_k, 2^{-k} + r) \ll_{q_b} (x, r)$ , we deduce that  $\Downarrow (x, r)$  is directed.

Finally, let  $(z, t)$  be an upper bound of  $\Downarrow (x, r)$ . Then  $q_b(x_n, z) \leq 2^{-n} + r - t$  for all  $n \in \mathbb{N}$ . Since  $q_b(x, z) = \inf_n \sup_{m \geq n} q_b(x_m, z)$ , we deduce that  $q_b(x, z) \leq r - t$ , and thus  $(x, r) \sqsubseteq_{q_b} (z, t)$ . Therefore  $(x, r)$  is the least upper bound of  $\Downarrow (x, r)$ .

We conclude that  $(\mathbf{B}\Sigma^\infty, \sqsubseteq_{q_b})$  is a domain.

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